Ch 6.2 Problems

Advanced Calculus
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Exercise 6.2.1: While uniform convergence preserves continuity, it does not preserve differentiability. Find an explicit example of a sequence of differentiable functions on $[-1, 1]$ that converge uniformly to a function f such that f is not differentiable.

Hint: There are many possibilities, simplest is perhaps to combine $|x|$ and $\frac{n}{2} \cdot x^2 + \frac{1}{2n}$, another is to consider $sqrtx + (1/n)$. Show that these functions are differentiable, converge uniformly, and then show that the limit is not differentiable.

$f_n \to funiformly \leftrightarrow \forall \varepsilon > 0, \exists N \epsilon Z^+ : \forall n > N$
$|f_n(x) - f(x)| < \varepsilon$
Work leading up to the proof:
$f_n(x) = \sqrt{x^2 - \frac{1}{n^2}}$
$f_n^{'}(x) = \frac{x}{\sqrt{x^2 - \frac{1}{n^2}}}$
$f_n^{'}(x) = 0$ when x=0
$f_n^{'}(0) = \sqrt{0^2 - (\frac{1}{2})^2} = \sqrt{\frac{-1}{n}} = \frac{i}{n}$
This is the maximum of the sequence (setting the derivative equal to zero and solving.
now, $|f_n(x) - 0| \leq \sqrt{\frac{-1}{n}} < \varepsilon$
$\frac{-1}{n} < \epsilon^2$
$\frac{-1}{\epsilon^2} < n$
this is the n value that we will need.

Proof:
let $\varepsilon > 0$ choose $N > \frac{-1}{\varepsilon^2}$

$\forall n > N, \forall x \epsilon R$
$|f_n(x) - 0| = \sqrt{x^2 - (\frac{1}{n})^2} \leq \sqrt{\frac{-1}{n}}$
since $n > N > \frac{1}{\varepsilon^2}$
so, $n > -1 \cdot \frac{1}{\varepsilon^2}$ $\sqrt{n} > \sqrt{\frac{-1}{\varepsilon^2}}$
$\sqrt{n} > \frac{\sqrt{-1}}{\varepsilon}$
$\varepsilon > \sqrt{\frac{-1}{n}}$
Thus, $|f_n(x) - 0| \leq \sqrt{\frac{-1}{n}} < \varepsilon$

Exercise 6.2.4: Show $lim_{n\to\infty} \int_1^2 e^{-nx^2} dx = 0$

$lim_{n\to\infty} \int_1^2 e^{-nx^2} dx = 0$
$lim_{n\to\infty} \frac{-2}{2n(2)} \cdot e^{-nx^2}$ from 1 to 2
$lim_{n\to\infty} \frac{-1}{2n(2)} \cdot e^{-n\cdot(2)^2} - (\frac{-1}{2n(1)} \cdot e^{-n\cdot 1^2})$
$lim_{n\to\infty} \frac{-1}{4n} \cdot e^{-4n} + \frac{1}{2n} \cdot e^{-n}$

as $n \to \infty$ $\frac{-1}{4n} \to 0$ and $\frac{1}{2n} \to 0$
recall that e^{-n} is equivalent to $\frac{1}{e^n}$ so as $n \to \infty$, $\frac{1}{e^n} \to \frac{1}{\infty} \to 0$ and $0 \cdot 0 = 0$ so this integral is true.

6.2.5 Find an example of a sequence of continuous functions on [0, 1] that converges pointwise to a continuous function on [0, 1], but the convergence is not uniform.

Recall $f_n \to f$ is pointwise if and only if $\varepsilon > 0$, $\forall x \epsilon [0, 1]$ $\exists N \epsilon Z^+$: $\forall n > N$ $|f_n(x) - f(x)| < \varepsilon$

Equation of choice: $f_n(x) = nxe^{-nx}$

Scratchwork:
$\varepsilon > 0, x \epsilon [0, 1]$
If x=0 $f_n(0) = 0$ then $|f_n(0) - 0| = |0 - 0| < \varepsilon$
If $x \neq 0$ then $|f_n(x) - 0| < \varepsilon$
$|nxe^{-nx}| < \varepsilon$, x is between zero and one so we do not need the absolute value anymore: $nxe^{-nx} < \varepsilon$
Recall the maclauren series: $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ we will compare this to what we have:

$e^{nx} > \frac{(nx)^2}{2!}$

$\frac{1}{e^{nx}} < \frac{2}{(nx)^2}$

$\frac{nx}{e^{nx}} = nx(\frac{1}{e^{nx}}) < nx \cdot \frac{2}{(nx)^2} = \frac{2}{nx} < \varepsilon$

now we solve for n: $\frac{2}{x} < n\varepsilon$

$\frac{2}{x\varepsilon} < n$
$n > \frac{2}{x\varepsilon}$

Now that this scratchwork is complete we can reformulate it in the reverse to make the proof:
let $\varepsilon > 0$ and $x\epsilon[0,1]$
we can choose an n such that $N > \frac{2}{x\varepsilon}$

Case 1: if x=0, then you can choose any $N\epsilon Z^+$ then $\forall n > N$ $|f_n(0) - 0| = |0 - 0| = 0 < \varepsilon$
in this case when x=0 the proof is done.

Case 2: when $x \neq 0$ choose $N > \frac{2}{x\varepsilon}$
then $\forall n > N$, $|f_n(x) - 0| = |nxe^{-nx}| = \frac{nx}{e^{nx}}$
$nx\frac{1}{e^{nx}} < \frac{2nx}{(nx)^2} = \frac{2}{nx}$
Since $n > N > \frac{2}{x\varepsilon}$ we now need to know that this is less than ε
$n > \frac{2}{x\varepsilon}$ so $\frac{x\varepsilon}{2} \cdot n > 1$
then $\frac{x\varepsilon}{2} > \frac{1}{n}$
so $\frac{1}{n} < \frac{x\varepsilon}{2}$
Thus $|f_n(x) - 0| = \frac{nx}{e^{nx}} < \frac{2}{nx} = \frac{1}{n} \cdot \frac{2}{x} < \frac{x\varepsilon}{2} \cdot \frac{2}{x} = \varepsilon$

This particular Problem the n vale depends on both ε and x which makes it converge solely pointwise and NOT uniform; recall the difference:

Converging Pointwise:

$f_n \to f$ pointwise if $\forall \varepsilon > 0$ and $\forall x\epsilon E$ then $\exists N\epsilon Z^+ : \forall n > N|f_n(x) - f(x)|$
Here N can depend on ε and, or x, therefore, it can converge at 2 different rates.

Uniform Convergence

$f_n \to f$ uniformly if $\forall \varepsilon > 0$ $\exists N\epsilon Z^+ : \forall n > N$ and $\forall x\epsilon E|f_n(x) - f(x)|$
Here N will only depend of ε thus, being uniform because the entire function depends solely on ε and converges all at the same rate.

Suppose $f_n : [a, b] \to R$ is a sequence of continuous functions that converges pointwise to a continuous $f : [a, b] \to R$. Suppose that for any $x \epsilon [a, b]$ the sequence $|f_n(x) f(x)|$ is monotone. Show that the sequence fn converges uniformly.

We need to find an n that satisifies the following:
$f_n \to f$ uniformly if $\forall \varepsilon > 0\ \exists N \epsilon Z^+ : \forall n > N$ and $\forall x \epsilon E |f_n(x) - f(x)|$