Chapter 6 Practice Problems

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6.1.1: Let f and g be bounded functions on [a,b], Prove: $\|f+g\|_u \leq \|f\|_u + \|g\|_u$

Proof: By squaring the left hand side you'll get:

$$(\|f+g\|_u)^2 = \|f+g\|_u \cdot \|f+g\|_u \tag{1a}$$

$$\|f\|_u^2 + \|2fg\|_u + \|g\|_u^2 \tag{1b}$$

Now by squaring the right hand side you'll get:

$$(\|f\|_u + \|g\|_u)^2 = (\|f\|_u + \|g\|_u) \cdot (\|f\|_u + \|g\|_u) \tag{1c}$$

$$\|f\|_u^2 + 2\,\|fg\|_u + \|g\|_u^2 \tag{1d}$$

This shows that the LHS is equal to the RHS, thus, if they are equal the inequality is true:

$$\|f\|_u^2 + \|2fg\|_u + \|g\|_u^2 = \|f\|_u^2 + 2\,\|fg\|_u + \|g\|_u^2 \tag{2a}$$

$$\|f+g\|_u \leq \|f\|_u + \|g\|_u \tag{2b}$$

6.1.2 (a) Find the pointwise $limit \frac{e^{x/n}}{n}$ for $x \epsilon R$

Fix x and let n go to infinity.
Case 1: $x > 0 as n \to \infty$
$x/n \to 0 as n \to \infty$
$e^0 = 1$
$\frac{1}{n} \to 0$

Case 2: $x < 0 n \to \infty$
The smaller the negative number is the exponential will approach zero.
$e^0 = 1$ $\frac{1}{n} \to 0$

Case 3: x=0
$e^0 = 1$

$\frac{1}{n} \to 0$

This function converges to zero for all of R. Therefore f exists and f(x)=0 for every $x \epsilon R$.

6.1.2 (b) Is the limit uniform on R

The series of functions here does not converge uniformly, since each f_n is unbounded.

6.1.2 (c) Is the limit uniform on [0,1]

Recall Proposition 6.1.13: Let fn : S → R be bounded functions. Then fn is Cauchy in the uniform norm if and only if there exists an f : S → R and fn converges uniformly to f .

Take any $\epsilon > 0$ and make $N > \frac{e}{\varepsilon}$
Then $\frac{e^{x/n}}{n} \leq \frac{e^{1/n}}{n}$
$\leq \frac{e^1}{n} \leq \frac{e}{n} < \epsilon$
It can be concluded that $f \to 0$ is uniform on [0,1]

6.1.9: Let fn : [0,1] → R be a sequence of increasing functions (that is, fn(x) ≥ fn(y) whenever x ≥ y). Suppose fn(0) = 0 and lim fn(1) = 0. Show that fn converges uniformly to 0. n