Chapter 3.2: Continuous Functions

Practice Problems

Emily West

Exercise 3.2.3: Let f : R → R be defined by f(x)= $\begin{cases} x & \text{if } x \text{ is rational} \\ x\text{^}2 & \text{if } x\text{^}2 \text{ is irratonal} \end{cases}$ prove it is continuous at 1 and not at 2.

Limit[$f(x)$, $x \to 1$] = Limit[x, x→x] = 1
Limit[$f(x)$, $x \to 1^-$] = Limit[x^2, x→1^-] = 1
Limit[$f(x)$, $x \to 1^+$] = Limit[x^2, x→1^+] = 1

Limit[$f(x)$, $x \to 2$] = Limit[x, x→2] = 2
Limit[$f(x)$, $x \to 2^-$] = Limit[x^2, x→2^-] = 4
Limit[$f(x)$, $x \to 2^+$] = Limit[x^2, x→2^+] = 4

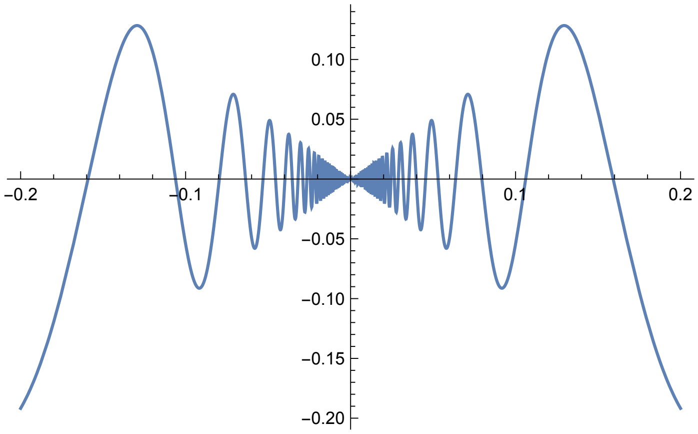
When approaching the rationals from either the positive or negative side of 1 you will still get 1, this is true for the irrationals, since 1^2 = 1. However, when approaching the rational number 2 on either side you'll get 2, but, for the irrationals when approaching 2^2 you'll always be approaching 4 on either side, not 2 making it not continuous at 2.

Exercise 3.2.4: Let f : R → R be defined by f(x):= $\begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ Is f continuous?

Lets approach zero from both the positive and negative side.
let, a_n= $\frac{1}{2 * \text{pi} * n}$ approach zero for n approaching infinity
then, f(a_n) = 0 for all n
and b_n= $\frac{1}{\frac{\text{pi}}{4} + 2 * \text{pi} * n}$ approach zero n approaching infinity
then, f(b_n) = $\frac{1}{\sqrt{2}}$ for all n
Since, a_n and b_n are approaching the same value but produce different values this is NOT continuous.

Exercise 3.2.5: Let f : R → R be defined by f(x):= $\begin{cases} x * \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ Is f continuous?

In[95]:= `Plot[x * Sin[1 / x], {x, -0.2, 0.2}]`

Out[95]=


0 ≤ | x*sin(1/x) |
= |x| |sin(1/x) |
≤ |x|
|sin(1/x) | ≤ |x|
Since sin(1/x) oscillates between 1 and -1 we will be focusing on how x affects this problem.
Think of sin(1/x) = $\frac{1}{x} - \frac{1}{x^3*3!} + \frac{1}{x^5*5!}$+... as a taylor series
then, xsin(1/x)=1- $\frac{1}{x^2*3!}$ + $\frac{1}{x^4*5!}$-... approaches 0 and x approaches infinity.
YES, this function is continuous.

Exercise 3.2.10: Let f : R → R and g : R → R be continuous functions. Suppose that for all rational numbers r, f (r) = g(r). Show that f (x) = g(x) for all x.

If we were to choose a rational number say $x_n$and from each interval we will get a rational sequence $\{x_n\}$ which converges to c.
Then let, f(x_n) =g(x_n) fro all n
by continuity of f(x) and g(x) we have:
Limit[$f(x_n)$, $n \to \infty$] = Limit[$f(x)$, $x \to c$] = f(c)
Limit[$g(x_n)$, $n \to \infty$] = Limit[$g(x)$, $x \to c$] = g(c)
therefore, f(c)= g(c) for every c that belongs to rationals.

Exercise 3.2.11: Let f : R → R be continuous. Suppose f (c) > 0. Show that there exists an α > 0 such that
for all x ∈ (c−α,c+α) we have f(x) > 0.

Proof by contradiction:
Suppose not, then for every positive integer n≥1 ∃ x_n with |x_n-c|< $\frac{1}{n}$and f(x_n) ≤ 0
Then Limit[x_n, $n \to \infty$]
and f is continuous so Limit[$f(x_n)$, $n \to \infty$] ≤ 0
but, Limit[$f(x_n)$, $n \to \infty$] > 0 therefore, this is a contradiction and the supposition is false.

Exercise 3.2.15: Suppose $g : R \to R$ is a continuous function such that $g(0) = 0$, and suppose $f : R \to R$ is such that $|f(x)- f(y)| \leq g(x-y)$ for all x and y. Show that f is continuous.

$g(x) \geq 0$ for every x

because $g(x) = g(x-c) \geq |f(x) - f(0)| \geq 0$

if $\epsilon > 0$ then $\exists\ \delta > 0 : g(z) \leq \epsilon$ if $|z| < \delta$

g is continuous at 0 and y(0) =0 where z=x-c

$g(x-c) \leq \epsilon$ of $|x-c| < \delta$

and then $|f(x)- f(c)| \leq g(x-c) \leq \epsilon$ if $|x-c| < \delta$

therefore, f is continuous.